Synchronization on complex networks
A model for neural networks

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What will I tell you today?

- a short introduction

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Synchronization on complex networks
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- a short introduction
- what a stochastic process is
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- synchronization: what, how and why?
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- synchronization: what, how and why?
- Kuramoto: a mathematical model
- synchronization on networks
Introduction

Processes on networks

1. Spreading of rumour
2. Searching for information on the internet
3. Formation of polymers
4. Synchronization of neurons firing in brain

What are some differences here?

1. Network as interactions or as paths
2. Process on each site or moving on network
3. Continuous space or discrete space
4. Dynamic or static network
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Examples of synchronization

1. Fireflies flashing in the jungle
2. Electricity generators on power grid
3. Audience clapping after concert
4. Neurons firing in the brain
5. Gravitational synchronization of meteors
6. and many more...

If you are looking for your next popular science book to read try:

'Sync: The emerging science of spontaneous order' - Steven Strogatz
Synchronization

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Question: What do you think?
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Ingredients

- some randomness

Example: Coin flipping

\( \omega = \{ H, T, T, T, H, T, H, ... \} \)
Question: What do you think?

Ingredients

- some *randomness*
- a recipe describing situation as *function of randomness*
Question: What do you think?

Ingredients

- some randomness
- a recipe describing situation as function of randomness
- some idea of time

Example: Coin flipping

win 1€ if heads
lose 1€ if tails
Question: What do you think?

Ingredients

- some randomness
- a recipe describing situation as function of randomness
- some idea of time

Example: Coin flipping

- win 1 € if heads
- lose 1 € if tails

Exercise:

$$\omega = \{H, T, T, T, H, T, H \ldots\}$$
Consider:

$$N$$ oscillators

$$\theta_i(t)$$ – phase of $$i$$th oscillator

Oscillators evolve according to a system of coupled stochastic differential equations

$$d\theta_i(t) = K \sum_{j=1}^{N} \sin[\theta_j(t) - \theta_i(t)] dt + D dW_i(t).$$

(1)

Here, $$K \in (0, \infty)$$ is the interaction strength, $$D \in (0, \infty)$$ is the noise strength, and $$(W_i(t))_{t \geq 0}$$ are noise processes.

Question: Can you spot the network here?
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Achtung! Mathematics ahead!
Consider:

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$$d\theta_i(t) = \frac{K}{N} \sum_{j=1}^{N} \sin [\theta_j(t) - \theta_i(t)] \, dt + D \, dW_i(t).$$

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Question:
Can you spot the network here?
Cartoon of the Kuramoto model for $N = 6$
Keeping track of the order

\[ r_N(t) = \sum_{j=1}^{N} e^{i \theta_j(t)}. \]  

\( r_N(t) \) – synchronization level
\( \psi_N(t) \) – average phase

Phase distributions with \( r = 0.095 \) and \( r = 0.929. \)
Keeping track of the order

**Order parameter**

\[ r_N(t) e^{i\psi_N(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}. \] (2)
Keeping track of the order

Order parameter

$$r_N(t) e^{i\psi_N(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)}.$$  \hspace{1cm} (2)

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- \( r_N(t) \) – synchronization level
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Phase distributions with \( r = 0.095 \) and \( r = 0.929 \).
Rewriting using the order parameter (exercise) gives

\[
d\theta_i(t) = Kr_N(t) \sin [\psi_N(t) - \theta_i(t)] \, dt + D \, dW_i(t),
\]  \hspace{1cm} (3)
Rewriting using the order parameter (exercise) gives

\[ \frac{d\theta_i(t)}{dt} = K r_N(t) \sin \left[ \psi_N(t) - \theta_i(t) \right] dt + D \, dW_i(t), \]  

(3)

The large \( N \) limit

As \( N \) gets ever larger, you can describe the evolution of the oscillators as the evolution of a density.
Rewriting using the order parameter (exercise) gives

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The large $N$ limit

As $N$ gets ever larger, you can describe the evolution of the oscillators as the evolution of a density.

But what is a density??
Rewriting using the order parameter (exercise) gives

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**The large N limit**

As $N$ gets ever larger, you can describe the evolution of the oscillators as the evolution of a density.

But what is a density??

**The large time limit (steady-state)**

Question: Does the density of the system stop evolving at some point?
There exists a critical threshold $K_c$ such that:

(I) For $K < K_c$ the system relaxes to an unsynchronized state ($r = 0$).

(II) For $K > K_c$ the system relaxes to a partially synchronized state ($r > 0$).
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**Theorem**

$$K_c = 2$$

(4)
| Suprachiasmatic nucleus |

SCN has a strong community structure, and the interaction between the communities is negative. This is the case in all mammals. The structure might play a role in the richness and robustness of the SCN. Malfunctioning can cause health problems ranging from epilepsy to narcolepsy.
Suprachiasmatic nucleus

- SCN has a strong community structure

Interaction between the communities is negative. This is the case in all mammals. Structure might play a role in the richness and robustness of SCN. Malfunctioning can cause health problems ranging from epilepsy to narcolepsy.
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Synchronization on complex networks

\[ r(K) = \frac{1}{2} \left( 1 + \cos(K(L=-2)) \right) \]
What should you remember?

There are many different types of processes to study on networks. Networks really play an important role almost everywhere. Synchronization is an example that is particularly interesting. Using mathematics we can tell neuroscientists something of value. Inter-disciplinary research is becoming more and more important.

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- Synchronization is an example that is particularly interesting.
Conclusion

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